Contents 16

Sequences and Series

Learning outcomes

In this Workbook you will learn about sequences and series. You will learn about arithmetic and geometric series and also about infinite series. You will learn how to test the for the convergence of an infinite series. You will then learn about power series, in particular you will study the binomial series. Finally you will apply your knowledge of power series to the process of finding series expansions of functions of a single variable. You will be able to find the Maclaurin and Taylor series expansions of simple functions about a point of interest.

Sequences and Series 16.1

Introduction

In this Section we develop the ground work for later Sections on infinite series and on power series. We begin with simple sequences of numbers and with finite series of numbers. We introduce the summation notation for the description of series. Finally, we consider arithmetic and geometric series and obtain expressions for the sum of n terms of both types of series.

Prerequisites

Before starting this Section you should ...

- understand and be able to use the basic rules of algebra
- be able to find limits of algebraic expressions

Learning Outcomes

On completion you should be able to ...

- check if a sequence of numbers is convergent
- use the summation notation to specify series
- recognise arithmetic and geometric series and find their sums

1. Introduction

A **sequence** is any succession of numbers. For example the sequence

$$1,\ 1,\ 2,\ 3,\ 5,\ 8,\ \ldots$$

which is known as the Fibonacci sequence, is formed by adding two consecutive terms together to obtain the next term. The numbers in this sequence continually increase without bound and we say this sequence **diverges**. An example of a **convergent** sequence is the **harmonic sequence**

$$1,\ \frac{1}{2},\ \frac{1}{3},\ \frac{1}{4},\ \ldots$$

Here we see the magnitude of these numbers continually decrease and it is obvious that the sequence converges to the number zero. The related **alternating harmonic sequence**

$$1,\ -\frac{1}{2},\ \frac{1}{3},\ -\frac{1}{4},\ \ldots$$

is also convergent to the number zero. Whether or not a sequence is convergent is often easy to deduce by graphing the individual terms. The diagrams in Figure 1 show how the individual terms of the harmonic and alternating harmonic series behave as the number of terms increase.

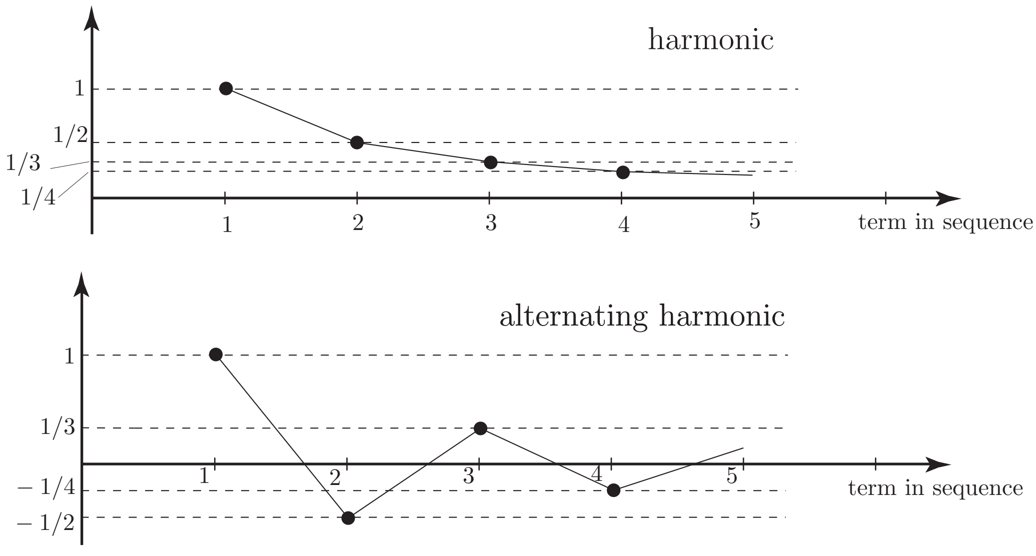


Figure 1

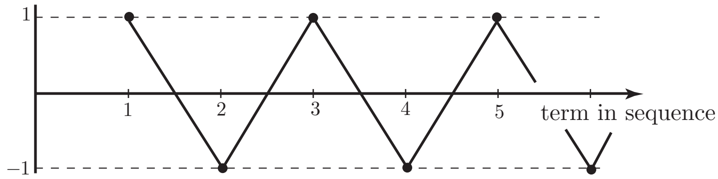
Graph the sequence:

$$1,\ -1,\ 1,\ -1,\ \ldots$$

Is this convergent?

Your solution

Answer



Not convergent.

The terms in the sequence do **not** converge to a particular value. The value **oscillates**.

A general sequence is denoted by

$$a_1,\ a_2,\ \ldots,\ a_n, \ldots$$

in which a_1 is the first term, a_2 is the second term and a_n is the n^{th} term is the sequence. For example, in the harmonic sequence

$$a_1 = 1, \quad a_2 = \frac{1}{2}, \ldots, a_n = \frac{1}{n}$$

whilst for the alternating harmonic sequence the n^{th} term is:

$$a_n = \frac{(-1)^{n+1}}{n}$$

in which $(-1)^n = +1$ if n is an even number and $(-1)^n = -1$ if n is an odd number.

Key Point 1

The sequence $a_1,\ a_2,\ \ldots, a_n,\ \ldots$ is said to be **convergent** if the limit of a_n as n increases can be found. (Mathematically we say that $\lim\limits_{n\to\infty} a_n$ exists.)

If the sequence is not convergent it is said to be **divergent**.

Task

Verify that the following sequence is convergent

$$\frac{3}{1\times 2},\ \frac{4}{2\times 3},\ \frac{5}{3\times 4}, \ \ldots$$

First find the expression for the n^{th} term:

Your solution

Answer

$$a_n = \frac{n+2}{n(n+1)}$$

Now find the limit of a_n as n increases:

Your solution

Answer

$$\frac{n+2}{n(n+1)} = \left[\frac{1+\dfrac{2}{n}}{n+1}\right] \rightarrow \frac{1}{n+1} \rightarrow 0 \qquad \text{as } n \text{ increases}$$

Hence the sequence is convergent.

2. Arithmetic and geometric progressions

Consider the sequences:

$$1,\ 4,\ 7,\ 10,\ \ldots \quad \text{and} \quad 3,\ 1,\ -1,\ -3,\ \ldots$$

In both, any particular term is obtained from the previous term by the **addition** of a constant value (3 and -2 respectively). Each of these sequences are said to be an **arithmetic sequence** or **arithmetic progression** and has general form:

$$a,\ a+d,\ a+2d,\ a+3d,\ \ldots,\ a+(n-1)d,\ \ldots$$

in which a, d are given numbers. In the first example above $a = 1,\ d = 3$ whereas, in the second example, $a = 3,\ d = -2$. The difference between any two successive terms of a given arithmetic sequence gives the value of d which is called the **common difference**.

Two sequences which are **not** arithmetic sequences are:

$$1,\ 2,\ 4,\ 8,\ \ldots$$

$$-1,\ -\frac{1}{3},\ -\frac{1}{9},\ -\frac{1}{27},\ \ldots$$

In each case a particular term is obtained from the previous term by **multiplying** by a constant factor (2 and $\frac{1}{3}$ respectively). Each is an example of a **geometric sequence** or **geometric progression** with the general form:

$$a,\ ar,\ ar^2,\ ar^3,\ \ldots$$

where 'a' is the first term and r is called the **common ratio**, being the ratio of two successive terms. In the first geometric sequence above $a = 1$, $r = 2$ and in the second geometric sequence $a = -1$, $r = \frac{1}{3}$.

Task Find a, d for the arithmetic sequence $3,\ 9,\ 15, \ldots$

Your solution

$a =$ $\qquad d =$

Answer

$a = 3, \quad d = 6$

Task Find a, r for the geometric sequence $8,\ \frac{8}{7},\ \frac{8}{49},\ \ldots$

Your solution

$a =$ $\qquad r =$

Answer

$a = 8, \quad r = \frac{1}{7}$

Task Write out the first four terms of the geometric series with $a = 4,\ r = -2$.

Your solution

Answer

$4, -8, 16, -32, \ldots$

The reader should note that many sequences (for example the harmonic sequences) are neither arithmetic nor geometric.

3. Series

A **series** is the sum of the terms of a sequence. For example, the **harmonic series** is

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots$$

and the **alternating harmonic series** is

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots$$

The summation notation

If we consider a general sequence

$$a_1,\ a_2,\ \ldots,\ a_n,\ \ldots$$

then the sum of the first k terms $\quad a_1+a_2+a_3+\cdots+a_k\quad$ is concisely denoted by $\displaystyle\sum_{p=1}^{k} a_p$.

That is,

$$a_1+a_2+a_3+\cdots+a_k=\sum_{p=1}^{k} a_p$$

When we encounter the expression $\displaystyle\sum_{p=1}^{k} a_p$ we let the index 'p' in the term a_p take, in turn, the values $1, 2, \ldots, k$ and then add all these terms together. So, for example

$$\sum_{p=1}^{3} a_p = a_1+a_2+a_3 \qquad\qquad \sum_{p=2}^{7} a_p = a_2+a_3+a_4+a_5+a_6+a_7$$

Note that p is a **dummy** index; any letter could be used as the index. For example $\displaystyle\sum_{i=1}^{6} a_i$, and $\displaystyle\sum_{m=1}^{6} a_m$ each represent the same collection of terms: $a_1+a_2+a_3+a_4+a_5+a_6$.

In order to be able to use this 'summation notation' we need to obtain a suitable expression for the 'typical term' in the series. For example, the finite series

$$1^2+2^2+\cdots+k^2$$

may be written as $\displaystyle\sum_{p=1}^{k} p^2$ since the typical term is clearly p^2 in which $p=1,2,3,\ldots,k$ in turn.

In the same way

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots-\frac{1}{16}=\sum_{p=1}^{16}\frac{(-1)^{p+1}}{p}$$

since an expression for the typical term in this alternating harmonic series is $a_p=\dfrac{(-1)^{p+1}}{p}$.

Write in summation form the series

$$\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\cdots+\frac{1}{21\times 22}$$

First find an expression for the typical term, "the p^{th} term":

Your solution

$a_p =$

Answer

$a_p = \dfrac{1}{p(p+1)}$

Now write the series in summation form:

Your solution

$$\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\cdots+\frac{1}{21\times 22}=$$

Answer

$$\frac{1}{1\times 2}+\frac{1}{2\times 3}+\cdots\qquad+\frac{1}{21\times 22}=\sum_{p=1}^{21}\frac{1}{p(p+1)}$$

Write out all the terms of the series $\displaystyle\sum_{p=1}^{5}\frac{(-1)^p}{(p+1)^2}$.

Give p the values $1, 2, 3, 4, 5$ in the typical term $\dfrac{(-1)^p}{(p+1)^2}$:

Your solution

$$\sum_{p=1}^{5}\frac{(-1)^p}{(p+1)^2}=$$

Answer

$$-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\frac{1}{5^2}-\frac{1}{6^2}.$$

4. Summing series

The arithmetic series

Consider the finite **arithmetic series** with 14 terms

$$1 + 3 + 5 + \cdots + 23 + 25 + 27$$

A simple way of working out the value of the sum is to create a second series which is the first written in reverse order. Thus we have two series, each with the same value A:

$$A = 1 + 3 + 5 + \cdots + 23 + 25 + 27$$

and

$$A = 27 + 25 + 23 + \cdots + 5 + 3 + 1$$

Now, adding the terms of these series in pairs

$$2A = 28 + 28 + 28 + \cdots + 28 + 28 + 28 = 28 \times 14 = 392 \qquad \text{so} \quad A = 196.$$

We can use this approach to find the sum of n terms of a general arithmetic series.
If

$$A = [a] + [a + d] + [a + 2d] + \cdots + [a + (n-2)d] + [a + (n-1)d]$$

then again simply writing the terms in reverse order:

$$A = [a + (n-1)d] + [a + (n-2)d] + \cdots + [a + 2d] + [a + d] + [a]$$

Adding these two identical equations together we have

$$2A = [2a + (n-1)d] + [2a + (n-1)d] + \cdots + [2a + (n-1)d]$$

That is, every one of the n terms on the right-hand side has the same value: $[2a + (n-1)d]$. Hence

$$2A = n[2a + (n-1)d] \qquad \text{so} \qquad A = \frac{1}{2}n[2a + (n-1)d].$$

Key Point 2

The **arithmetic series**

$$[a] + [a + d] + [a + 2d] + \cdots + [a + (n-1)d]$$

having n terms has sum A where:

$$A = \frac{1}{2}n[2a + (n-1)d]$$

As an example

$$1 + 3 + 5 + \cdots + 27 \quad \text{has} \quad a = 1,\ d = 2,\ n = 14$$

So $A = 1 + 3 + \cdots + 27 = \dfrac{14}{2}[2 + (13)2] = 196.$

The geometric series

We can also sum a general **geometric series**.
Let

$$G = a + ar + ar^2 + \cdots + ar^{n-1}$$

be a geometric series having exactly n terms. To obtain the value of G in a more convenient form we first multiply through by the common ratio r:

$$rG = ar + ar^2 + ar^3 + \cdots + ar^n$$

Now, writing the two series together:

$$G = a + ar + ar^2 + \cdots + ar^{n-1}$$

$$rG = ar + ar^2 + ar^3 + \cdots ar^{n-1} + ar^n$$

Subtracting the second expression from the first we see that all terms on the right-hand side cancel out, except for the first term of the first expression and the last term of the second expression so that

$$G - rG = (1-r)G = a - ar^n$$

Hence (assuming $r \neq 1$)

$$G = \frac{a(1-r^n)}{1-r}$$

(Of course, if $r = 1$ the geometric series simplifies to a simple arithmetic series with $d = 0$ and has sum $G = na$.)

Key Point 3

The **geometric series**

$$a + ar + ar^2 + \cdots + ar^{n-1}$$

having n terms has sum G where

$$G = \frac{a(1-r^n)}{1-r}, \quad \text{if } r \neq 1 \qquad \text{and} \qquad G = na, \quad \text{if } r = 1$$

Task

Find the sum of each of the following series:

(a) $1 + 2 + 3 + 4 + \cdots + 100$

(b) $\dfrac{1}{2} + \dfrac{1}{6} + \dfrac{1}{18} + \dfrac{1}{54} + \dfrac{1}{162} + \dfrac{1}{486}$

(a) In this arithmetic series state the values of a, d, n:

Your solution

$a =$ $\qquad d =$ $\qquad n =$

Answer

$a = 1, \quad d = 1, \quad n = 100.$

Now find the sum:

Your solution

$1 + 2 + 3 + \cdots + 100 =$

Answer

$1 + 2 + 3 + \cdots + 100 = 50(2 + 99) = 50(101) = 5050.$

(b) In this geometric series state the values of a, r, n:

Your solution

$a =$ $\qquad r =$ $\qquad n =$

Answer

$a = \dfrac{1}{2}, \quad r = \dfrac{1}{3}, \quad n = 6$

Now find the sum:

Your solution

$\dfrac{1}{2} + \dfrac{1}{6} + \dfrac{1}{18} + \dfrac{1}{54} + \dfrac{1}{162} + \dfrac{1}{486} =$

Answer

$$\frac{1}{2} + \frac{1}{6} + \cdots + \frac{1}{486} = \frac{1}{2}\frac{\left(1 - \left(\frac{1}{3}\right)^6\right)}{1 - \frac{1}{3}} = \frac{3}{4}\left(1 - \left(\frac{1}{3}\right)^6\right) = 0.74897$$

Exercises

1. Which of the following sequences is convergent?

(a) $\sin\dfrac{\pi}{2},\ \sin\dfrac{2\pi}{2},\ \sin\dfrac{3\pi}{2},\ \sin\dfrac{4\pi}{2},\ \ldots$

(b) $\dfrac{\sin\frac{\pi}{2}}{\frac{\pi}{2}},\ \dfrac{\sin\frac{2\pi}{2}}{\frac{2\pi}{2}},\ \dfrac{\sin\frac{3\pi}{2}}{\frac{3\pi}{2}},\ \dfrac{\sin\frac{4\pi}{2}}{\frac{4\pi}{2}},\ \ldots$

2. Write the following series in summation form:

(a) $\dfrac{\ln 1}{2\times 1}+\dfrac{\ln 3}{3\times 2}+\dfrac{\ln 5}{4\times 3}+\cdots+\dfrac{\ln 27}{15\times 14}$

(b) $-\dfrac{1}{2\times(1+(100)^2)}+\dfrac{1}{3\times(1-(200)^2)}-\dfrac{1}{4\times(1+(300)^2)}+\cdots+\dfrac{1}{9\times(1-(800)^2)}$

3. Write out the first three terms and the last term of the following series:

(a) $\displaystyle\sum_{p=1}^{17}\frac{3^{p-1}}{p!(18-p)}$ (b) $\displaystyle\sum_{p=4}^{17}\frac{(-p)^{p+1}}{p(2+p)}$

4. Sum the series:

(a) $-5\ \ -1\ \ +3\ \ +7\ \ \ldots\ \ +27$

(b) $-5\ \ -9\ \ -13\ \ -17\ \ \ldots\ \ -37$

(c) $\dfrac{1}{2}-\dfrac{1}{6}+\dfrac{1}{18}-\dfrac{1}{54}+\dfrac{1}{162}-\dfrac{1}{486}$

Answers

1. (a) no; this sequence is $1,\ 0,\ -1,\ 0,\ 1,\ldots$ which does not converge.

(b) yes; this sequence is $\dfrac{1}{\pi/2},\ 0,\ -\dfrac{1}{3\pi/2},\ 0,\ \dfrac{1}{5\pi/2},\ldots$ which converges to zero.

2. (a) $\displaystyle\sum_{p=1}^{14}\frac{\ln(2p-1)}{(p+1)(p)}$ (b) $\displaystyle\sum_{p=1}^{8}\frac{(-1)^p}{(p+1)(1+(-1)^{p+1}p^2 10^4)}$

3. (a) $\dfrac{1}{17},\ \dfrac{3}{2!(16)},\ \dfrac{3^2}{3!(15)},\ \ldots\ ,\dfrac{3^{16}}{17!}$ (b) $-\dfrac{4^5}{(4)(6)},\ \dfrac{5^6}{(5)(7)},\ -\dfrac{6^7}{(6)(8)},\ \ldots\ ,\dfrac{17^{18}}{(17)(19)}$

4. (a) This is an arithmetic series with $a=-5,\ d=4,\ n=9.\quad A=99$

(b) This is an arithmetic series with $a=-5,\ d=-4,\ n=9.\quad A=-189$

(c) This is a geometric series with $a=\dfrac{1}{2},\ r=-\dfrac{1}{3},\ n=6.\quad G\approx 0.3745$

Infinite Series

16.2

Introduction

We extend the concept of a finite series, met in Section 16.1, to the situation in which the number of terms increase without bound. We define what is meant by an infinite series being **convergent** by considering the **partial sums** of the series. As prime examples of infinite series we examine the harmonic and the alternating harmonic series and show that the former is divergent and the latter is convergent.

We consider various tests for the convergence of series, in particular we introduce the ratio test which is a test applicable to series of positive terms. Finally we define the meaning of the terms absolute and conditional convergence.

Prerequisites

Before starting this Section you should ...

- be able to use the $\sum$ summation notation
- be familiar with the properties of limits
- be able to use inequalities

Learning Outcomes

On completion you should be able to ...

- use the alternating series test on infinite series
- use the ratio test on infinite series
- understand the terms absolute and conditional convergence

1. Introduction

Many of the series considered in Section 16.1 were examples of **finite series** in that they all involved the summation of a finite number of terms. When the number of terms in the series increases without bound we refer to the sum as an **infinite series**. Of particular concern with infinite series is whether they are convergent or divergent. For example, the infinite series

$$1 + 1 + 1 + 1 + \cdots$$

is clearly divergent because the sum of the first n terms increases without bound as more and more terms are taken. It is less clear as to whether the harmonic and alternating harmonic series:

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots \qquad\qquad 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

converge or diverge. Indeed you may be surprised to find that the first is divergent and the second is convergent. What we shall do in this Section is to consider some simple convergence tests for infinite series. Although we all have an intuitive idea as to the meaning of convergence of an infinite series we must be more precise in our approach. We need a definition for convergence which we can apply rigorously.

First, using an obvious extension of the notation we have used for a finite sum of terms, we denote the infinite series:

$$a_1 + a_2 + a_3 + \cdots + a_p + \cdots \qquad \text{by the expression} \quad \sum_{p=1}^{\infty} a_p$$

where a_p is an expression for the p^{th} term in the series. So, as examples:

$$\begin{aligned}
1 + 2 + 3 + \cdots &= \sum_{p=1}^{\infty} p \qquad \text{since the } p^{th} \text{ term is } a_p \equiv p \\
1^2 + 2^2 + 3^2 + \cdots &= \sum_{p=1}^{\infty} p^2 \qquad \text{since the } p^{th} \text{ term is } a_p \equiv p^2 \\
1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots &= \sum_{p=1}^{\infty} \frac{(-1)^{p+1}}{p} \qquad \text{here} \qquad a_p \equiv \frac{(-1)^{p+1}}{p}
\end{aligned}$$

Consider the infinite series:

$$a_1 + a_2 + \cdots + a_p + \cdots = \sum_{p=1}^{\infty} a_p$$

We consider the **sequence of partial sums**, $S_1, S_2, \ldots,$ of this series where

$$\begin{aligned}
S_1 &= a_1 \\
S_2 &= a_1 + a_2 \\
&\vdots \\
S_n &= a_1 + a_2 + \cdots + a_n
\end{aligned}$$

That is, S_n is the sum of the first n terms of the infinite series. If the limit of the sequence $S_1,\ S_2, \ldots, S_n, \ldots$ can be found; that is

$$\lim_{n\to\infty} S_n = S \qquad \text{(say)}$$

then we define the sum of the infinite series to be S:

$$S = \sum_{p=1}^{\infty} a_p$$

and we say "the series converges to S". Another way of stating this is to say that

$$\sum_{p=1}^{\infty} a_p = \lim_{n\to\infty} \sum_{p=1}^{n} a_p$$

Key Point 4

Convergence of Infinite Series

An infinite series $\sum_{p=1}^{\infty} a_p$ is convergent if the sequence of partial sums

$$S_1,\ S_2,\ S_3,\ \ldots, S_k, \ldots \qquad \text{in which} \quad S_k = \sum_{p=1}^{k} a_p \quad \text{is convergent}$$

Divergence condition for an infinite series

An almost obvious requirement that an infinite series should be convergent is that the individual terms in the series should get smaller and smaller. This leads to the following Key Point:

Key Point 5

The condition:

$$a_p \to 0 \quad \text{as} \quad p \text{ increases} \quad (\text{mathematically} \quad \lim_{p\to\infty} a_p = 0)$$

is a **necessary** condition for the convergence of the series $\sum_{p=1}^{\infty} a_p$

It is not possible for an infinite series to be convergent unless this condition holds.

Which of the following series cannot be convergent?

(a) $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \cdots$

(b) $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$

(c) $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$

In each case, use the condition from Key Point 5:

Your solution

(a) $a_p =$ $\qquad\qquad$ $\lim_{p\to\infty} a_p =$

Answer

$a_p = \dfrac{p}{p+1} \qquad \lim_{p\to\infty} \dfrac{p}{p+1} = 1$

Hence series is divergent.

Your solution

(b) $a_p =$ $\qquad\qquad$ $\lim_{p\to\infty} a_p =$

Answer

$a_p = \dfrac{1}{p} \qquad \lim_{p\to\infty} a_p = 0$

So this series **may** be convergent. Whether it is or not requires further testing.

Your solution

(c) $a_p =$ $\qquad\qquad$ $\lim_{p\to\infty} a_p =$

Answer

$a_p = \dfrac{(-1)^{p+1}}{p} \qquad \lim_{p\to\infty} a_p = 0$ so again this series **may** be convergent.

Divergence of the harmonic series

The harmonic series:

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$$

has a general term $a_n = \dfrac{1}{n}$ which clearly gets smaller and smaller as $n \to \infty$. However, surprisingly, the series is divergent. Its divergence is demonstrated by showing that the harmonic series is greater than another series which is obviously divergent. We do this by grouping the terms of the harmonic series in a particular way:

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots \quad \equiv \quad 1 + \left(\frac{1}{2}\right) + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \cdots$$

Now

$$\begin{aligned}
\left(\frac{1}{3}+\frac{1}{4}\right) &> \frac{1}{4}+\frac{1}{4}=\frac{1}{2}\\
\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right) &> \frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}=\frac{1}{2}\\
\left(\frac{1}{9}+\frac{1}{10}+\frac{1}{11}+\frac{1}{12}+\frac{1}{13}+\frac{1}{14}+\frac{1}{15}+\frac{1}{16}\right) &> \frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}=\frac{1}{2}
\end{aligned}$$

and so on. Hence the harmonic series satisfies:

$$1+\left(\frac{1}{2}\right)+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)+\cdots > 1+\left(\frac{1}{2}\right)+\left(\frac{1}{2}\right)+\left(\frac{1}{2}\right)+\cdots$$

The right-hand side of this inequality is clearly divergent so **the harmonic series is divergent.**

Convergence of the alternating harmonic series

As with the harmonic series we shall group the terms of the alternating harmonic series, this time to display its convergence.
The alternating harmonic series is:

$$S=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\cdots$$

This series may be re-grouped in two distinct ways.

1st re-grouping

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}+\cdots=1-\left(\frac{1}{2}-\frac{1}{3}\right)-\left(\frac{1}{4}-\frac{1}{5}\right)-\left(\frac{1}{6}-\frac{1}{7}\right)\cdots$$

each term in brackets is positive since $\frac{1}{2}>\frac{1}{3}$, $\frac{1}{4}>\frac{1}{5}$ and so on. So we easily conclude that $S<1$ since we are subtracting only positive numbers from 1.

2nd re-grouping

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}+\cdots=\left(1-\frac{1}{2}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\left(\frac{1}{5}-\frac{1}{6}\right)+\cdots$$

Again, each term in brackets is positive since $1>\frac{1}{2}$, $\frac{1}{3}>\frac{1}{4}$, $\frac{1}{5}>\frac{1}{6}$ and so on.

So we can also argue that $S>\frac{1}{2}$ since we are adding only positive numbers to the value of the first term, $\frac{1}{2}$. The conclusion that is forced upon us is that

$$\frac{1}{2}<S<1$$

so the alternating series is convergent since its sum, S, lies in the range $\frac{1}{2}\to 1$. It will be shown in Section 16.5 that $S=\ln 2\simeq 0.693$.

2. General tests for convergence

The techniques we have applied to analyse the harmonic and the alternating harmonic series are 'one-off':- they cannot be applied to infinite series in general. However, there are many tests that can be used to determine the convergence properties of infinite series. Of the large number available we shall only consider two such tests in detail.

The alternating series test

An alternating series is a special type of series in which the sign changes from one term to the next. They have the form

$$a_1 - a_2 + a_3 - a_4 + \cdots$$

(in which each $a_i,\ i = 1, 2, 3, \ldots$ is a **positive** number)
Examples are:

(a) $1 - 1 + 1 - 1 + 1 \cdots$

(b) $\frac{1}{3} - \frac{2}{4} + \frac{3}{5} - \frac{4}{6} + \cdots$

(c) $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$.

For series of this type there is a simple criterion for convergence:

Key Point 6

The Alternating Series Test

The alternating series

$$a_1 - a_2 + a_3 - a_4 + \cdots$$

(in which each $a_i,\ i = 1, 2, 3, \ldots$ are **positive** numbers) is convergent if and only if

- the terms continually decrease:

$$a_1 > a_2 > a_3 > \ldots$$

- the terms decrease to zero:

$$a_p \to 0 \quad \text{as } p \text{ increases} \qquad (\text{mathematically } \lim_{p\to\infty} a_p = 0)$$

Task

Which of the following series are convergent?

(a) $\displaystyle\sum_{p=1}^{\infty}(-1)^p\frac{(2p-1)}{(2p+1)}$ (b) $\displaystyle\sum_{p=1}^{\infty}\frac{(-1)^{p+1}}{p^2}$

(a) First, write out the series:

Your solution

Answer

$-\frac{1}{3}+\frac{3}{5}-\frac{5}{7}+\cdots$

Now examine the series for convergence:

Your solution

Answer

$\dfrac{(2p-1)}{(2p+1)}=\dfrac{(1-\frac{1}{2p})}{(1+\frac{1}{2p})}\to 1$ as p increases.

Since the individual terms of the series do not converge to zero this is therefore a divergent series.

(b) Apply the procedure used in (a) to problem (b):

Your solution

Answer

This series $1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\cdots$ is an alternating series of the form $a_1-a_2+a_3-a_4+\cdots$ in which $a_p=\frac{1}{p^2}$. The a_p sequence is a decreasing sequence since $1>\frac{1}{2^2}>\frac{1}{3^2}>\ldots$

Also $\displaystyle\lim_{p\to\infty}\frac{1}{p^2}=0$. Hence the series is convergent by the alternating series test.

3. The ratio test

This test, which is one of the most useful and widely used convergence tests, applies only to series of **positive terms.**

Key Point 7

The Ratio Test

Let $\displaystyle\sum_{p=1}^{\infty} a_p$ be a series of **positive** terms such that, as p increases, the limit of $\dfrac{a_{p+1}}{a_p}$ equals a number λ. That is $\displaystyle\lim_{p\to\infty} \frac{a_{p+1}}{a_p} = \lambda$.

It can be shown that:

- if $\lambda > 1$, then $\displaystyle\sum_{p=1}^{\infty} a_p$ diverges
- if $\lambda < 1$, then $\displaystyle\sum_{p=1}^{\infty} a_p$ converges
- if $\lambda = 1$, then $\displaystyle\sum_{p=1}^{\infty} a_p$ may converge or diverge.

That is, the test is inconclusive in this case.

Example 1

Use the ratio test to examine the convergence of the series

(a) $1+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\cdots$ (b) $1+x+x^2+x^3+\cdots$

Solution

(a) The general term in this series is $\frac{1}{p!}$ i.e.

$$1+\frac{1}{2!}+\frac{1}{3!}+\cdots=\sum_{p=1}^{\infty}\frac{1}{p!} \qquad a_p=\frac{1}{p!} \qquad \therefore \qquad a_{p+1}=\frac{1}{(p+1)!}$$

and the ratio

$$\frac{a_{p+1}}{a_p}=\frac{p!}{(p+1)!}=\frac{p(p-1)\ldots(3)(2)(1)}{(p+1)p(p-1)\ldots(3)(2)(1)}=\frac{1}{(p+1)}$$

$$\therefore \qquad \lim_{p\to\infty}\frac{a_{p+1}}{a_p}=\lim_{p\to\infty}\frac{1}{(p+1)}=0$$

Since $0<1$ the series is convergent. In fact, it will be easily shown, using the techniques outlined in HELM 16.5, that

$$1+\frac{1}{2!}+\frac{1}{3!}+\cdots=\mathrm{e}-1\approx 1.718$$

(b) Here we must assume that $x>0$ since we can only apply the ratio test to a series of positive terms.

Now

$$1+x+x^2+x^3+\cdots \quad =\sum_{p=1}^{\infty}x^{p-1}$$

so that

$$a_p=x^{p-1}\ , \quad a_{p+1}=x^p$$

and

$$\lim_{p\to\infty}\frac{a_{p+1}}{a_p}=\lim_{p\to\infty}\frac{x^p}{x^{p-1}}=\lim_{p\to\infty}x=x$$

Thus, using the ratio test we deduce that (if x is a positive number) this series will only converge if $x<1$.

We will see in Section 16.4 that

$$1+x+x^2+x^3+\cdots \quad =\frac{1}{1-x} \qquad \text{provided} \quad 0<x<1.$$

Task

Use the ratio test to examine the convergence of the series:

$$\frac{1}{\ln 3}+\frac{8}{(\ln 3)^2}+\frac{27}{(\ln 3)^3}+\cdots$$

First, find the general term of the series:

Your solution

$a_p =$

Answer

$$\frac{1}{\ln 3}+\frac{8}{(\ln 3)^2}+\cdots=\sum_{p=1}^{\infty}\frac{p^3}{(\ln 3)^p} \quad \text{so} \quad a_p=\frac{p^3}{(\ln 3)^p}$$

Now find a_{p+1}:

Your solution

$a_{p+1} =$

Answer

$$a_{p+1}=\frac{(p+1)^3}{(\ln 3)^{p+1}}$$

Finally, obtain $\displaystyle\lim_{p\to\infty}\frac{a_{p+1}}{a_p}$:

Your solution

$$\frac{a_{p+1}}{a_p}= \qquad\qquad \therefore \qquad \lim_{p\to\infty}\frac{a_{p+1}}{a_p}=$$

Answer

$$\frac{a_{p+1}}{a_p}=\left(\frac{p+1}{p}\right)^3\frac{1}{(\ln 3)}. \text{ Now } \left(\frac{p+1}{p}\right)^3=\left(1+\frac{1}{p}\right)^3\to 1 \text{ as } p \text{ increases}$$

$$\therefore \qquad \lim_{p\to\infty}\frac{a_{p+1}}{a_p}=\frac{1}{(\ln 3)}<1$$

Hence this is a convergent series.

Note that in all of these Examples and Tasks we have decided upon the convergence or divergence of various series; we have not been able to use the tests to discover what actual number the convergent series converges to.

4. Absolute and conditional convergence

The ratio test applies to series of positive terms. Indeed this is true of many related tests for convergence. However, as we have seen, not all series are series of positive terms. To apply the ratio test such series must first be converted into series of positive terms. This is easily done. Consider two series $\sum_{p=1}^{\infty} a_p$ and $\sum_{p=1}^{\infty} |a_p|$. The latter series, obviously directly related to the first, is a series of positive terms.

Using imprecise language, it is harder for the second series to converge than it is for the first, since, in the first, some of the terms may be negative and cancel out part of the contribution from the positive terms. No such cancellations can take place in the second series since they are all positive terms. Thus it is plausible that if $\sum_{p=1}^{\infty} |a_p|$ converges so does $\sum_{p=1}^{\infty} a_p$. This leads to the following definitions.

Key Point 8

Conditional Convergence and Absolute Convergence

A convergent series $\sum_{p=1}^{\infty} a_p$ is said to be **conditionally convergent** if $\sum_{p=1}^{\infty} |a_p|$ is divergent.

A convergent series $\sum_{p=1}^{\infty} a_p$ is said to be **absolutely convergent** if $\sum_{p=1}^{\infty} |a_p|$ is convergent.

For example, the alternating harmonic series:

$$\sum_{p=1}^{\infty} \frac{(-1)^{p+1}}{p} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

is **conditionally convergent** since the series of positive terms (the harmonic series):

$$\sum_{p=1}^{\infty} \left| \frac{(-1)^{p+1}}{p} \right| \equiv \sum_{p=1}^{\infty} \frac{1}{p} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$$

is divergent.

Task Show that the series $-\dfrac{1}{2!}+\dfrac{1}{4!}-\dfrac{1}{6!}+\cdots$ is absolutely convergent.

First, find the general term of the series:

Your solution

$$-\frac{1}{2!}+\frac{1}{4!}-\frac{1}{6!}+\cdots \quad =\sum_{p=1}^{\infty}(\qquad) \qquad \therefore \qquad a_p \equiv$$

Answer

$$-\frac{1}{2!}+\frac{1}{4!}-\frac{1}{6!}+\cdots \quad =\sum_{p=1}^{\infty}\frac{(-1)^p}{(2p)!} \qquad \therefore \qquad a_p \equiv \frac{(-1)^p}{(2p)!}$$

Write down an expression for the related series of positive terms:

Your solution

$$\frac{1}{2!}+\frac{1}{4!}+\frac{1}{6!}+\cdots \quad =\sum_{p=1}^{\infty}(\qquad\qquad) \qquad \therefore \qquad a_p =$$

Answer

$$\sum_{p=1}^{\infty}\frac{1}{(2p)!} \qquad \text{so} \qquad a_p=\frac{1}{(2p)!}$$

Now use the ratio test to examine the convergence of this series:

Your solution

p^{th} term $=$ $\qquad\qquad$ $(p+1)^{th}$ term $=$

Answer

p^{th} term $=\dfrac{1}{(2p)!}$ $\qquad\qquad$ $(p+1)^{th}$ term $=\dfrac{1}{(2(p+1))!}$

Find $\displaystyle\lim_{p\to\infty}\left[\frac{(p+1)^{th}\text{ term}}{p^{th}\text{ term}}\right]$:

Your solution

$$\lim_{p\to\infty}\left[\frac{(p+1)^{th}\text{ term}}{p^{th}\text{ term}}\right]=$$

Answer

$$\frac{(2p)!}{(2(p+1))!}=\frac{2p(2p-1)\ldots}{(2p+2)(2p+1)2p(2p-1)\ldots}=\frac{1}{(2p+2)(2p+1)}\to 0 \text{ as } p \text{ increases.}$$

So the series of positive terms is convergent by the ratio test. Hence $\displaystyle\sum_{p=1}^{\infty}\frac{(-1)^p}{(2p)!}$ is absolutely convergent.

Exercises

1. Which of the following alternating series are convergent?

(a) $\displaystyle\sum_{p=1}^{\infty}\frac{(-1)^p\ln(3)}{p}$ (b) $\displaystyle\sum_{p=1}^{\infty}\frac{(-1)^{p+1}}{p^2+1}$ (c) $\displaystyle\sum_{p=1}^{\infty}\frac{p\sin(2p+1)\frac{\pi}{2}}{(p+100)}$

2. Use the ratio test to examine the convergence of the series:

(a) $\displaystyle\sum_{p=1}^{\infty}\frac{e^4}{(2p+1)^{p+1}}$ (b) $\displaystyle\sum_{p=1}^{\infty}\frac{p^3}{p!}$ (c) $\displaystyle\sum_{p=1}^{\infty}\frac{1}{\sqrt{p}}$

(d) $\displaystyle\sum_{p=1}^{\infty}\frac{1}{(0.3)^p}$ (e) $\displaystyle\sum_{p=1}^{\infty}\frac{(-1)^{p+1}}{3^p}$

3. For what values of x are the following series absolutely convergent?

(a) $\displaystyle\sum_{p=1}^{\infty}\frac{(-1)^p x^p}{p}$ (b) $\displaystyle\sum_{p=1}^{\infty}\frac{(-1)^p x^p}{p!}$

Answers

1. (a) convergent, (b) convergent, (c) divergent

2. (a) $\lambda = 0$ so convergent

   (b) $\lambda = 0$ so convergent

   (c) $\lambda = 1$ so test is inconclusive. However, since $\dfrac{1}{p^{1/2}} > \dfrac{1}{p}$ then the given series is divergent by comparison with the harmonic series.

   (d) $\lambda = 10/3$ so divergent, (e) Not a series of positive terms so the ratio test cannot be applied.

3. (a) The related series of positive terms is $\displaystyle\sum_{p=1}^{\infty}\frac{|x|^p}{p}$. For this series, using the ratio test we find $\lambda = |x|$ so the original series is absolutely convergent if $|x| < 1$.

   (b) The related series of positive terms is $\displaystyle\sum_{p=1}^{\infty}\frac{|x|^p}{p!}$. For this series, using the ratio test we find $\lambda = 0$ (irrespective of the value of x) so the original series is absolutely convergent for **all** values of x.

The Binomial Series

16.3

Introduction

In this Section we examine an important example of an infinite series, the **binomial** series:

$$1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \cdots$$

We show that this series is only convergent if $|x| < 1$ and that in this case the series sums to the value $(1+x)^p$. As a special case of the binomial series we consider the situation when p is a positive integer n. In this case the infinite series reduces to a **finite** series and we obtain, by replacing x with $\frac{b}{a}$, the **binomial theorem**:

$$(b+a)^n = b^n + nb^{n-1}a + \frac{n(n-1)}{2!}b^{n-2}a^2 + \cdots + a^n.$$

Finally, we use the binomial series to obtain various polynomial expressions for $(1+x)^p$ when x is 'small'.

Prerequisites

Before starting this Section you should ...

- understand the factorial notation
- have knowledge of the ratio test for convergence of infinite series.
- understand the use of inequalities

Learning Outcomes

On completion you should be able to ...

- recognise and use the binomial series
- state and use the binomial theorem
- use the binomial series to obtain numerical approximations

1. The binomial series

A very important infinite series which occurs often in applications and in algebra has the form:

$$1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \cdots$$

in which p is a given number and x is a variable. By using the ratio test it can be shown that this series converges, irrespective of the value of p, as long as $|x| < 1$. In fact, as we shall see in Section 16.5 the given series converges to the value $(1+x)^p$ as long as $|x| < 1$.

Key Point 9

The Binomial Series

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \cdots \qquad |x| < 1$$

The binomial theorem can be obtained directly from the binomial series if p is chosen to be a **positive integer** (here we need not demand that $|x| < 1$ as the series is now finite and so is always convergent irrespective of the value of x). For example, with $p = 2$ we obtain

$$\begin{aligned}(1+x)^2 &= 1 + 2x + \frac{2(1)}{2}x^2 + 0 + 0 + \cdots \\ &= 1 + 2x + x^2 \quad \text{as is well known.}\end{aligned}$$

With $p = 3$ we get

$$\begin{aligned}(1+x)^3 &= 1 + 3x + \frac{3(2)}{2}x^2 + \frac{3(2)(1)}{3!}x^3 + 0 + 0 + \cdots \\ &= 1 + 3x + 3x^2 + x^3\end{aligned}$$

Generally if $p = n$ (a positive integer) then

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots + x^n$$

which is a form of the binomial theorem. If x is replaced by $\dfrac{b}{a}$ then

$$\left(1 + \frac{b}{a}\right)^n = 1 + n\left(\frac{b}{a}\right) + \frac{n(n-1)}{2!}\left(\frac{b}{a}\right)^2 + \cdots + \left(\frac{b}{a}\right)^n$$

Now multiplying both sides by a^n we have the following Key Point:

Key Point 10

The Binomial Theorem

If n is a positive integer then the expansion of $(a+b)$ raised to the power n is given by:

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \cdots + b^n$$

This is known as the **binomial** theorem.

Task

Use the binomial theorem to obtain (a) $(1+x)^7$ (b) $(a+b)^4$

(a) Here $n=7$:

Your solution

$(1+x)^7 =$

Answer

$(1+x)^7 = 1 + 7x + 21x^2 + 35x^3 + 35x^4 + 21x^5 + 7x^6 + x^7$

(b) Here $n=4$:

Your solution

$(a+b)^4 =$

Answer

$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$.

Task

Given that x is so small that powers of x^3 and above may be ignored in comparison to lower order terms, find a quadratic approximation of $(1-x)^{\frac{1}{2}}$ and check for accuracy your approximation for $x = 0.1$.

First expand $(1-x)^{\frac{1}{2}}$ using the binomial series with $p = \dfrac{1}{2}$ and with x replaced by $(-x)$:

Your solution

$(1-x)^{\frac{1}{2}} =$

Answer

$(1-x)^{\frac{1}{2}} = 1 - \frac{1}{2}x + \frac{\frac{1}{2}(-\frac{1}{2})}{2}x^2 - \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{6}x^3 + \cdots$

Now obtain the quadratic approximation:

Your solution

$$(1-x)^{\frac{1}{2}} \simeq$$

Answer

$(1-x)^{\frac{1}{2}} \simeq 1 - \dfrac{1}{2}x - \dfrac{1}{8}x^2$

Now check on the validity of the approximation by choosing $x = 0.1$:

Your solution

Answer

On the left-hand side we have

$$(0.9)^{\frac{1}{2}} = 0.94868 \text{ to } 5 \text{ d.p.} \qquad \text{obtained by calculator}$$

whereas, using the quadratic expansion:

$$(0.9)^{\frac{1}{2}} \approx 1 - \frac{1}{2}(0.1) - \frac{1}{8}(0.1)^2 = 1 - 0.05 - (0.00125) = 0.94875.$$

so the error is only 0.00007.

What we have done in this last Task is to replace (or approximate) the function $(1-x)^{\frac{1}{2}}$ by the simpler (polynomial) function $1 - \dfrac{1}{2}x - \dfrac{1}{8}x^2$ which is reasonable provided x is very small. This approximation is well illustrated geometrically by drawing the curves $y = (1-x)^{\frac{1}{2}}$ and $y = 1 - \dfrac{1}{2}x - \dfrac{1}{8}x^2$. The two curves coincide when x is 'small'. See Figure 2:

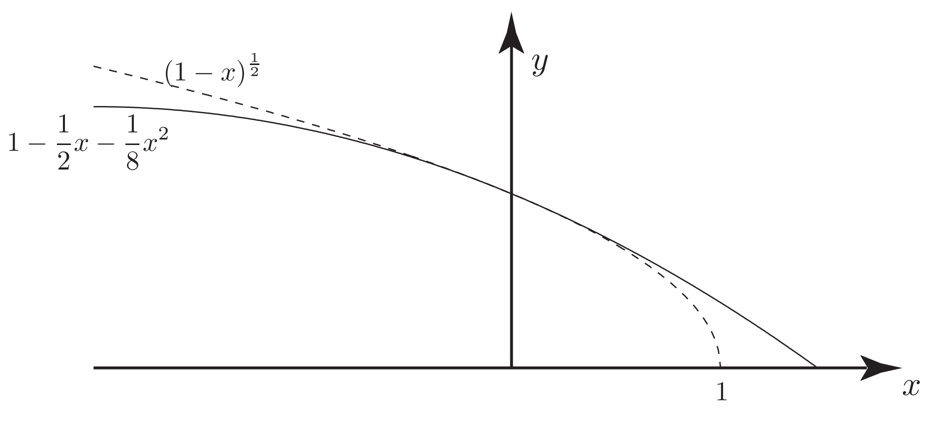


Figure 2

Task

Obtain a cubic approximation of $\dfrac{1}{(2+x)}$. Check your approximation for accuracy using appropriate values of x.

First write the term $\dfrac{1}{(2+x)}$ in a form suitable for the binomial series (refer to Key Point 9):

Your solution

$$\frac{1}{(2+x)} =$$

Answer

$$\frac{1}{2+x} = \frac{1}{2\left(1+\dfrac{x}{2}\right)} = \frac{1}{2}\left(1+\frac{x}{2}\right)^{-1}$$

Now expand using the binomial series with $p = -1$ and $\dfrac{x}{2}$ instead of x, to include terms up to x^3:

Your solution

$$\frac{1}{2}\left(1+\frac{x}{2}\right)^{-1} =$$

Answer

$$\begin{aligned}\frac{1}{2}\left(1+\frac{x}{2}\right)^{-1} &= \frac{1}{2}\left\{1+(-1)\frac{x}{2}+\frac{(-1)(-2)}{2!}\left(\frac{x}{2}\right)^2+\frac{(-1)(-2)(-3)}{3!}\left(\frac{x}{2}\right)^3\right\}\\ &= \frac{1}{2}-\frac{x}{4}+\frac{x^2}{8}-\frac{x^3}{16}\end{aligned}$$

State the range of x for which the binomial series of $\left(1+\dfrac{x}{2}\right)^{-1}$ is valid:

Your solution

The series is valid if

Answer

valid as long as $\left|\dfrac{x}{2}\right| < 1$ i.e. $|x| < 2$ or $-2 < x < 2$

Choose $x = 0.1$ to check the accuracy of your approximation:

Your solution

$$\frac{1}{2}\left(1+\frac{0.1}{2}\right)^{-1} =$$

$$\frac{1}{2} - \frac{0.1}{4} + \frac{0.01}{8} - \frac{0.001}{16} =$$

Answer

$\frac{1}{2}\left(1+\frac{0.1}{2}\right)^{-1} = 0.47619$ to 5 d.p.

$\frac{1}{2} - \frac{0.1}{4} + \frac{0.01}{8} - \frac{0.001}{16} = 0.4761875.$

Figure 3 below illustrates the close correspondence (when x is 'small') between the curves $y = \frac{1}{2}(1+\frac{x}{2})^{-1}$ and $y = \frac{1}{2} - \frac{x}{4} + \frac{x^2}{8} - \frac{x^3}{16}$.

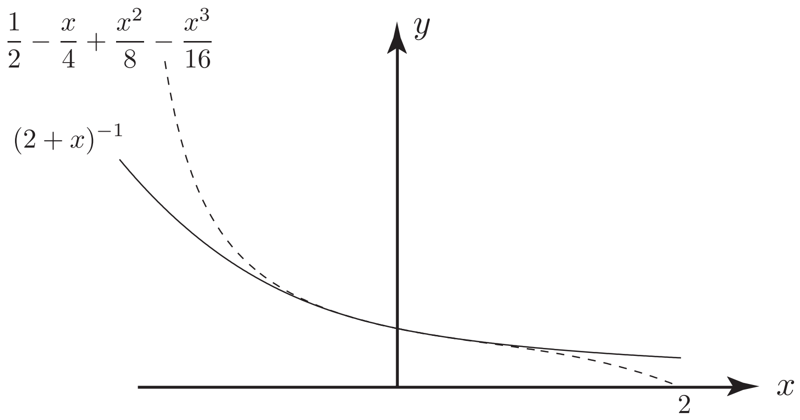


Figure 3

Exercises

1. Determine the expansion of each of the following

(a) $(a+b)^3$, (b) $(1-x)^5$, (c) $(1+x^2)^{-1}$, (d) $(1-x)^{1/3}$.

2. Obtain a cubic approximation (valid if x is small) of the function $(1+2x)^{3/2}$.

Answers

1. (a) $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

(b) $(1-x)^5 = 1 - 5x + 10x^2 - 10x^3 + 5x^4 - x^5$

(c) $(1+x^2)^{-1} = 1 - x^2 + x^4 - x^6 + \cdots$

(d) $(1-x)^{1/3} = 1 - \frac{1}{3}x - \frac{1}{9}x^2 - \frac{5}{81}x^3 + \cdots$

2. $(1+2x)^{3/2} = 1 + 3x + \frac{3}{2}x^2 - \frac{1}{2}x^3 + \cdots$

Power Series

16.4

Introduction

In this Section we consider power series. These are examples of infinite series where each term contains a variable, x, raised to a positive integer power. We use the ratio test to obtain the **radius of convergence** R, of the power series and state the important result that the series is absolutely convergent if $|x| < R$, divergent if $|x| > R$ and may or may not be convergent if $x = \pm R$. Finally, we extend the work to apply to general power series when the variable x is replaced by $(x - x_0)$.

Prerequisites

Before starting this Section you should ...

- have knowledge of infinite series and of the ratio test
- have knowledge of inequalities and of the factorial notation.

Learning Outcomes

On completion you should be able to ...

- explain what a power series is
- obtain the radius of convergence for a power series
- explain what a general power series is

1. Power series

A power series is simply a sum of terms each of which contains a variable raised to a non-negative integer power. To illustrate:

$$x - x^3 + x^5 - x^7 + \cdots$$

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

are examples of power series. In HELM 16.3 we encountered an important example of a power series, the binomial series:

$$1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \cdots$$

which, as we have already noted, represents the function $(1+x)^p$ as long as the variable x satisfies $|x| < 1$.

A power series has the general form

$$b_0 + b_1 x + b_2 x^2 + \cdots = \sum_{p=0}^{\infty} b_p x^p$$

where $b_0, b_1, b_2, \cdots$ are constants. Note that, in the summation notation, we have chosen to start the series at $p = 0$. This is to ensure that the power series can include a constant term b_0 since $x^0 = 1$.

The convergence, or otherwise, of a power series, clearly depends upon the value of x chosen. For example, the power series

$$1 + \frac{x}{2} + \frac{x^2}{3} + \frac{x^3}{4} + \cdots$$

is convergent if $x = -1$ (for then it is the alternating harmonic series) and divergent if $x = +1$ (for then it is the harmonic series).

2. The radius of convergence

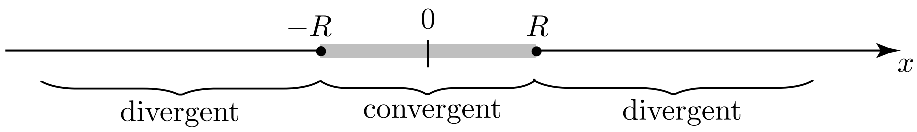
The most important statement one can make about a power series is that there exists a number, R, called the radius of convergence, such that if $|x| < R$ the power series is absolutely convergent and if $|x| > R$ the power series is divergent. At the two points $x = -R$ and $x = R$ the power series may be convergent or divergent.

Key Point 11

Convergence of Power Series

For a power series $\sum_{p=0}^{\infty} b_p x^p$ with radius of convergence R then

- the series converges absolutely if $|x| < R$
- the series diverges if $|x| > R$
- the series may be convergent or divergent at $x = \pm R$



For any particular power series $\sum_{p=0}^{\infty} b_p x^p$ the value of R can be obtained using the ratio test. We know, from the ratio test that $\sum_{p=0}^{\infty} b_p x^p$ is absolutely convergent if

$$\lim_{p\to\infty} \frac{|b_{p+1}x^{p+1}|}{|b_p x^p|} = \lim_{p\to\infty} \left|\frac{b_{p+1}}{b_p}\right| |x| < 1 \quad \text{implying} \quad |x| < \lim_{p\to\infty} \left|\frac{b_p}{b_{p+1}}\right| \quad \text{and so} \quad R = \lim_{p\to\infty} \left|\frac{b_p}{b_{p+1}}\right|.$$

Example 2

(a) Find the radius of convergence of the series

$$1 + \frac{x}{2} + \frac{x^2}{3} + \frac{x^3}{4} + \cdots$$

(b) Investigate what happens at the end-points $x = -1$, $x = +1$ of the region of absolute convergence.

Solution

(a) Here $1+\frac{x}{2}+\frac{x^2}{3}+\frac{x^3}{4}+\cdots \quad = \sum_{p=0}^{\infty}\frac{x^p}{p+1}$

so

$$b_p = \frac{1}{p+1} \qquad \therefore \qquad b_{p+1} = \frac{1}{p+2}$$

In this case,

$$R = \lim_{p\to\infty}\left|\frac{p+2}{p+1}\right| = 1$$

so the given series is absolutely convergent if $|x| < 1$ and is divergent if $|x| > 1$.

(b) At $x = +1$ the series is $1+\frac{1}{2}+\frac{1}{3}+\cdots$ which is divergent (the harmonic series). However, at $x = -1$ the series is $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots$ which is convergent (the alternating harmonic series).

Finally, therefore, the series

$$1+\frac{x}{2}+\frac{x^2}{3}+\frac{x^3}{4}+\cdots$$

is convergent if $-1 \leq x < 1$.

Task

Find the range of values of x for which the following power series converges:

$$1+\frac{x}{3}+\frac{x^2}{3^2}+\frac{x^3}{3^3}+\cdots$$

First find the coefficient of x^p:

Your solution

$b_p =$

Answer

$b_p = \frac{1}{3^p}$

Now find R, the radius of convergence:

Your solution

$$R = \lim_{p\to\infty}\left|\frac{b_p}{b_{p+1}}\right| =$$

Answer

$$R = \lim_{p\to\infty}\left|\frac{b_p}{b_{p+1}}\right| = \lim_{p\to\infty}\left|\frac{3^{p+1}}{3^p}\right| = \lim_{p\to\infty}(3) = 3.$$

When $x = \pm 3$ the series is clearly divergent. Hence the series is convergent only if $-3 < x < 3$.

3. Properties of power series

Let P_1 and P_2 represent two power series with radii of convergence R_1 and R_2 respectively. We can combine P_1 and P_2 together by addition and multiplication. We find the following properties:

Key Point 12

If P_1 and P_2 are power series with respective radii of convergence R_1 and R_2 then the sum (P_1+P_2) and the product (P_1P_2) are each power series with the radius of convergence being the **smaller** of R_1 and R_2.

Power series can also be differentiated and integrated on a term by term basis:

Key Point 13

If P_1 is a power series with radius of convergence R_1 then

$$\frac{d}{dx}(P_1) \quad \text{and} \quad \int (P_1)\,dx$$

are each power series with radius of convergence R_1

Example 3

Using the known result that $\quad (1+x)^p = 1 + px + \dfrac{p(p-1)}{2!}x^2 + \cdots \qquad |x| < 1,$

choose $p = \frac{1}{2}$ and by differentiating obtain the power series expression for $(1+x)^{-\frac{1}{2}}$.

Solution

$$(1+x)^{\frac{1}{2}} = 1 + \frac{x}{2} + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2!}x^2 + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}x^3 + \cdots$$

Differentiating both sides: $\quad \dfrac{1}{2}(1+x)^{-\frac{1}{2}} = \dfrac{1}{2} + \dfrac{1}{2}\left(-\dfrac{1}{2}\right)x + \dfrac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2}x^2 + \cdots$

Multiplying through by 2: $\quad (1+x)^{-\frac{1}{2}} = 1 - \dfrac{1}{2}x + \dfrac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2}x^2 + \cdots$

This result can, of course, be obtained directly from the expansion for $(1+x)^p$ with $p = -\frac{1}{2}$.

Task

Using the known result that

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots \qquad |x| < 1,$$

(a) Find an expression for $\ln(1+x)$

(b) Use the expression to obtain an approximation to $\ln(1.1)$

(a) Integrate both sides of $\dfrac{1}{1+x} = 1 - x + x^2 - \cdots$ and so deduce an expression for $\ln(1+x)$:

Your solution

$$\int \frac{dx}{1+x} =$$

$$\int (1 - x + x^2 - \cdots)\, dx =$$

Answer

$\displaystyle\int \frac{dx}{1+x} = \ln(1+x) + c$ where c is a constant of integration,

$\displaystyle\int (1 - x + x^2 - \cdots)\, dx = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + k$ where k is a constant of integration.

So we conclude $\qquad \displaystyle\ln(1+x) + c = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + k \qquad$ if $\quad |x| < 1$

Choosing $x = 0$ shows that $c = k$ so they cancel from this equation.

(b) Now choose $x = 0.1$ to approximate $\ln(1 + 0.1)$ using terms up to cubic:

Your solution

$$\ln(1.1) = 0.1 - \frac{(0.1)^2}{2} + \frac{(0.1)^3}{3} - \cdots \simeq$$

Answer

$\ln(1.1) \simeq 0.0953$ which is easily checked by calculator.

4. General power series

A general power series has the form

$$b_0 + b_1(x - x_0) + b_2(x - x_0)^2 + \cdots = \sum_{p=0}^{\infty} b_p(x - x_0)^p$$

Exactly the same considerations apply to this general power series as apply to the 'special' series $\sum_{p=0}^{\infty} b_p x^p$ except that the variable x is replaced by $(x - x_0)$. The radius of convergence of the general series is obtained in the same way:

$$R = \lim_{p\to\infty} \left| \frac{b_p}{b_{p+1}} \right|$$

and the interval of convergence is now shifted to have centre at $x = x_0$ (see Figure 4 below). The series is absolutely convergent if $|x - x_0| < R$, diverges if $|x - x_0| > R$ and may or may not converge if $|x - x_0| = R$.

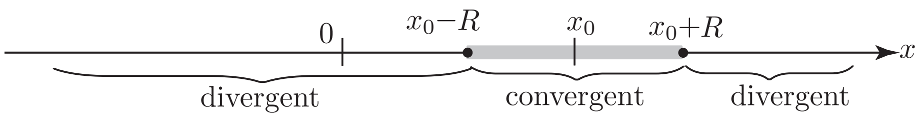


Figure 4

Task

Find the radius of convergence of the general power series

$$1 - (x - 1) + (x - 1)^2 - (x - 1)^3 + \cdots$$

First find an expression for the general term:

Your solution

$$1 - (x - 1) + (x - 1)^2 - (x - 1)^3 + \cdots = \sum_{p=0}^{\infty}$$

Answer

$\sum_{p=0}^{\infty}(x - 1)^p(-1)^p$ so $b_p = (-1)^p$

Now obtain the radius of convergence:

Your solution

$\lim_{p\to\infty} \left| \frac{b_p}{b_{p+1}} \right| =$ $\therefore$ $R =$

Answer

$\lim_{p\to\infty} \left| \frac{b_p}{b_{p+1}} \right| = \lim_{p\to\infty} \left| \frac{(-1)^p}{(-1)^{p+1}} \right| = 1.$

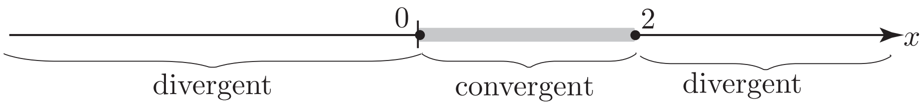
Hence $R = 1$, so the series is absolutely convergent if $|x - 1| < 1$.

Finally, decide on the convergence at $|x-1| = 1$ (i.e. at $x-1=-1$ and $x-1=1$ i.e. $x=0$ and $x=2$):

Your solution

Answer

At $x=0$ the series is $1+1+1+\cdots$ which diverges and at $x=2$ the series is $1-1+1-1\cdots$ which also diverges. Thus the given series only converges if $|x-1|<1$ i.e. $0<x<2$.



Exercises

1. From the result $\dfrac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots\ , \quad |x| < 1$

   (a) Find an expression for $\ln(1-x)$

   (b) Use this expression to obtain an approximation to $\ln(0.9)$ to 4 d.p.

2. Find the radius of convergence of the general power series $1-(x+2)+(x+2)^2-(x+2)^3+\ldots$

3. Find the range of values of x for which the power series $1+\dfrac{x}{4}+\dfrac{x^2}{4^2}+\dfrac{x^3}{4^3}+\ldots$ converges.

4. By differentiating the series for $(1+x)^{1/3}$ find the power series for $(1+x)^{-2/3}$ and state its radius of convergence.

5. (a) Find the radius of convergence of the series $1+\dfrac{x}{3}+\dfrac{x^2}{4}+\dfrac{x^3}{5}+\ldots$

   (b) Investigate what happens at the points $x=-1$ and $x=+1$

Answers

1. $\ln(1-x) = -x - \dfrac{x^2}{2} - \dfrac{x^3}{3} - \dfrac{x^4}{4} - \ldots \qquad \ln(0.9) \approx -0.1054$ (4 d.p.)

2. $R=1$. Series converges if $-3<x<-1$. If $x=-1$ series diverges. If $x=-3$ series diverges.

3. Series converges if $-4<x<4$.

4. $(1+x)^{-2/3} = 1 - \dfrac{2}{3}x + \dfrac{5}{3}x^2 + \ldots$ valid for $|x|<1$.

5. (a) $R=1$. (b) At $x=+1$ series diverges. At $x=-1$ series converges.

Maclaurin and Taylor Series

16.5

Introduction

In this Section we examine how functions may be expressed in terms of power series. This is an extremely useful way of expressing a function since (as we shall see) we can then replace 'complicated' functions in terms of 'simple' polynomials. The only requirement (of any significance) is that the 'complicated' function should be *smooth*; this means that at a point of interest, it must be possible to differentiate the function as often as we please.

Prerequisites

Before starting this Section you should ...

- have knowledge of power series and of the ratio test
- be able to differentiate simple functions
- be familiar with the rules for combining power series

Learning Outcomes

On completion you should be able to ...

- find the Maclaurin and Taylor series expansions of given functions
- find Maclaurin expansions of functions by combining known power series together
- find Maclaurin expansions by using differentiation and integration

1. Maclaurin and Taylor series

As we shall see, many functions can be represented by power series. In fact we have already seen in earlier Sections examples of such a representation:

$$\begin{aligned}
\frac{1}{1-x} &= 1+x+x^2+\cdots \qquad |x|<1 \\
\ln(1+x) &= x-\frac{x^2}{2}+\frac{x^3}{3}-\cdots \qquad -1<x\le 1 \\
e^x &= 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots \qquad \text{all } x
\end{aligned}$$

The first two examples show that, as long as we constrain x to lie within the domain $|x|<1$ (or, equivalently, $-1<x<1$), then in the first case $\frac{1}{1-x}$ has the **same numerical value** as $1+x+x^2+\cdots$ and in the second case $\ln(1+x)$ has the same numerical value as $x-\frac{x^2}{2}+\frac{x^3}{3}-\cdots$. In the third example we see that e^x has the same numerical value as $1+x+\frac{x^2}{2!}+\cdots$ but in this case there is no restriction to be placed on the value of x since this power series converges for all values of x. Figure 5 shows this situation geometrically. As more and more terms are used from the series $1+x+\frac{x^2}{2!}+\frac{x^3}{3!}\cdots$ the curve representing e^x is a better and better approximation. In (a) we show the linear approximation to e^x. In (b) and (c) we show, respectively, the quadratic and cubic approximations.

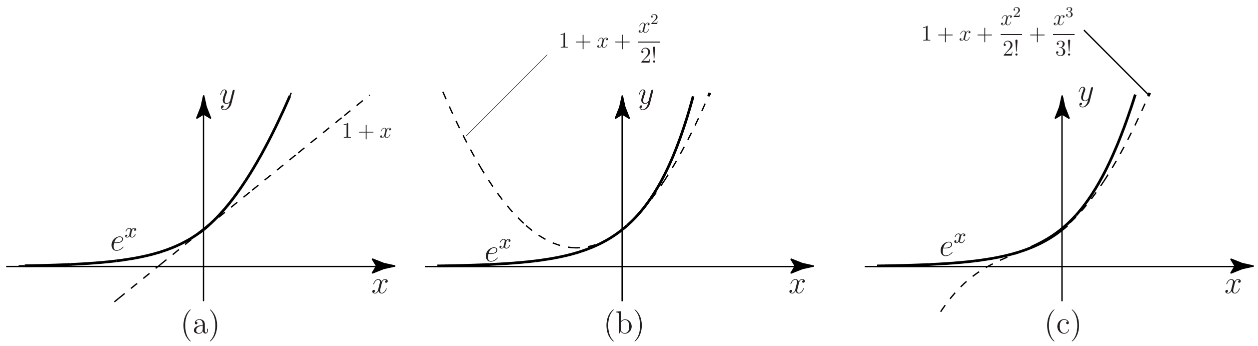


Figure 5: Linear, quadratic and cubic approximations to e^x

These power series representations are extremely important, from many points of view. Numerically, we can simply replace the function $\frac{1}{1-x}$ by the quadratic expression $1+x+x^2$ as long as x is so small that powers of x greater than or equal to 3 can be ignored in comparison to quadratic terms. This approach can be used to approximate more complicated functions in terms of simpler polynomials. Our aim now is to see how these power series expansions are obtained.

2. The Maclaurin series

Consider a function $f(x)$ which can be differentiated at $x = 0$ as often as we please. For example $e^x, \cos x, \sin x$ would fit into this category but $|x|$ would not.
Let us assume that $f(x)$ can be represented by a power series in x:

$$f(x) = b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + \cdots = \sum_{p=0}^{\infty} b_p x^p$$

where $b_0, b_1, b_2, \ldots$ are constants to be determined.

If we substitute $x = 0$ then, clearly $\quad f(0) = b_0$

The other constants can be determined by further differentiating and, on each differentiation, substituting $x = 0$. For example, differentiating once:

$$f'(x) = 0 + b_1 + 2b_2 x + 3b_3 x^2 + 4b_4 x^3 + \cdots$$

so, putting $x = 0$, we have $f'(0) = b_1$.
Continuing to differentiate:

$$f''(x) = 0 + 2b_2 + 3(2)b_3 x + 4(3)b_4 x^2 + \cdots$$

so

$$f''(0) = 2b_2 \qquad \text{or} \qquad b_2 = \frac{1}{2} f''(0)$$

Further:

$$f'''(x) = 3(2)b_3 + 4(3)(2)b_4 x + \cdots \qquad \text{so} \qquad f'''(0) = 3(2)b_3 \qquad \text{implying} \qquad b_3 = \frac{1}{3(2)} f'''(0)$$

Continuing in this way we easily find that (remembering that $0! = 1$)

$$b_n = \frac{1}{n!} f^{(n)}(0) \qquad n = 0, 1, 2, \ldots$$

where $f^{(n)}(0)$ means the value of the n^{th} derivative at $x = 0$ and $f^{(0)}(0)$ means $f(0)$.
Bringing all these results together we have:

Key Point 14

Maclaurin Series

If $f(x)$ can be differentiated as often as required:

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots = \sum_{p=0}^{\infty} \frac{x^p}{p!} f^{(p)}(0)$$

This is called the **Maclaurin expansion** of $f(x)$.

Example 4

Find the Maclaurin expansion of $\cos x$.

Solution

Here $f(x) = \cos x$ and, differentiating a number of times:

$$f(x) = \cos x, \qquad f'(x) = -\sin x, \qquad f''(x) = -\cos x, \qquad f'''(x) = \sin x \quad \text{etc.}$$

Evaluating each of these at $x = 0$:

$$f(0) = 1, \quad f'(0) = 0, \quad f''(0) = -1, \quad f'''(0) = 0 \text{ etc.}$$

Substituting into $f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots$, gives:

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

The reader should confirm (by finding the radius of convergence) that this series is convergent for **all** values of x. The geometrical approximation to $\cos x$ by the first few terms of its Maclaurin series are shown in Figure 6.

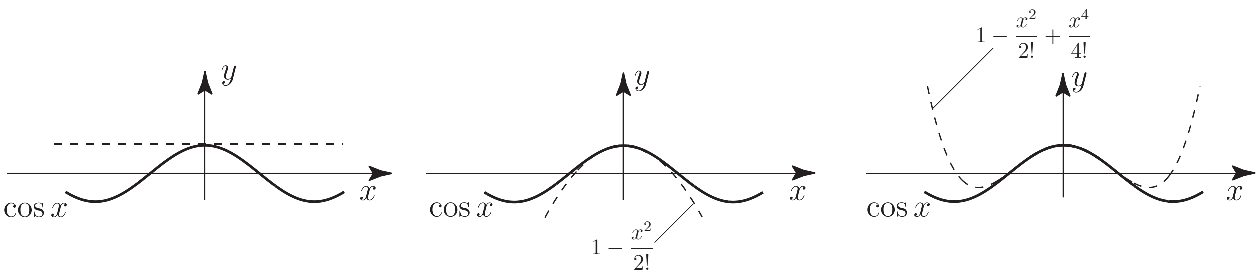


Figure 6: Linear, quadratic and cubic approximations to $\cos x$

Task

Find the Maclaurin expansion of $\ln(1 + x)$.

(Note that we **cannot** find a Maclaurin expansion of the function $\ln x$ since $\ln x$ does not exist at $x = 0$ and so cannot be differentiated at $x = 0$.)

Find the first four derivatives of $f(x) = \ln(1 + x)$:

Your solution

$f'(x) =$ $\qquad$ $f''(x) =$ $\qquad$ $f'''(x) =$ $\qquad$ $f''''(x) =$

Answer

$$f'(x) = \frac{1}{1+x}, \qquad f''(x) = \frac{-1}{(1+x)^2}, \qquad f'''(x) = \frac{2}{(1+x)^3},$$

generally: $\quad f^{(n)}(x) = \dfrac{(-1)^{n+1}(n-1)!}{(1+x)^n}$

Now obtain $f(0),\ f'(0),\ f''(0),\ f'''(0)$:

Your solution

$f(0) = \qquad f'(0) = \qquad f''(0) = \qquad f'''(0) =$

Answer

$f(0) = 0 \qquad f'(0) = 1, \qquad f''(0) = -1, \qquad f'''(0) = 2,$

generally: $\quad f^{(n)}(0) = (-1)^{n+1}(n-1)!$

Hence, obtain the Maclaurin expansion of $\ln(1+x)$:

Your solution

$\ln(1+x) =$

Answer

$\ln(1+x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} \ldots + \dfrac{(-1)^{n+1}}{n}x^n + \cdots$ (This was obtained in Section 16.4, page 37.)

Now obtain the radius of convergence and consider the situation at the boundary values:

Your solution

Radius of convergence $\qquad R =$

Answer

$R = 1$. Also at $x = 1$ the series is convergent (alternating harmonic series) and at $x = -1$ the series is divergent. Hence this Maclaurin expansion is only valid if $-1 < x \leq 1$.

The geometrical closeness of the polynomial terms with the function $\ln(1+x)$ for $-1 < x \leq 1$ is displayed in Figure 7:

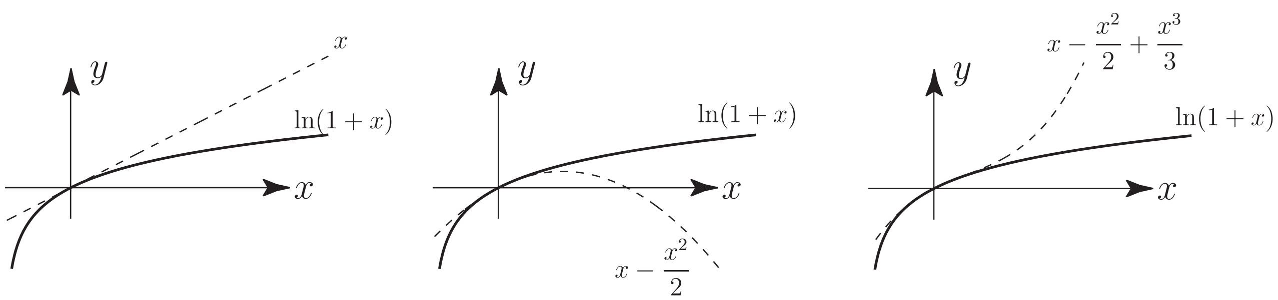


Figure 7: Linear, quadratic and cubic approximations to $\ln(1+x)$

Note that when $x = 1$ $\quad \ln 2 = 1 - \dfrac{1}{2} + \dfrac{1}{3} - \dfrac{1}{4} \cdots$ so the alternating harmonic series converges to $\ln 2 \simeq 0.693$, as stated in Section 16.2, page 17.

The Maclaurin expansion of a product of two functions: $f(x)g(x)$ is obtained by multiplying together the Maclaurin expansions of $f(x)$ and of $g(x)$ and collecting like terms together. The product series will have a radius of convergence equal to the **smaller** of the two separate radii of convergence.

Example 5

Find the Maclaurin expansion of $e^x \ln(1+x)$.

Solution

Here, instead of finding the derivatives of $f(x) = e^x \ln(1+x)$, we can more simply multiply together the Maclaurin expansions for e^x and $\ln(1+x)$ which we already know:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \qquad \text{all } x$$

and

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots \qquad -1 < x \leq 1$$

The resulting power series will only be convergent if $-1 < x \leq 1$. Multiplying:

$$\begin{aligned} e^x \ln(1+x) &= \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)\left(x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots\right) \\ &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \\ &\qquad + x^2 - \frac{x^3}{2} + \frac{x^4}{3} + \cdots \\ &\qquad\qquad + \frac{x^3}{2} - \frac{x^4}{4} \cdots \\ &\qquad\qquad\qquad + \frac{x^4}{6} \cdots \\ &= x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{3x^5}{40} + \cdots \qquad -1 < x \leq 1 \end{aligned}$$

(You must take care not to miss relevant terms when carrying through the multiplication.)

Find the Maclaurin expansion of $\cos^2 x$ up to powers of x^4. Hence write down the expansion of $\sin^2 x$ to powers of x^6.

First, write down the expansion of $\cos x$:

Your solution

$\cos x =$

Answer

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$$

Now, by multiplication, find the expansion of $\cos^2 x$:

Your solution

$\cos^2 x =$

Answer

$$\cos^2 x = \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} \cdots\right)\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} \cdots\right)$$

$$= (1 - \frac{x^2}{2!} + \frac{x^4}{4!} \cdots) + (-\frac{x^2}{2!} + \frac{x^4}{4} \cdots) + (\frac{x^4}{4!} \cdots) + \cdots = 1 - x^2 + \frac{x^4}{3} - \frac{2x^6}{45} \cdots$$

Now obtain the expansion of $\sin^2 x$ using a suitable trigonometric identity:

Your solution

$\sin^2 x =$

Answer

$$\sin^2 x = 1 - \cos^2 x = 1 - \left(1 - x^2 + \frac{x^4}{3} - \frac{2x^6}{45} + \cdots\right) = x^2 - \frac{x^4}{3} + \frac{2x^6}{45} + \cdots$$

As an alternative approach the reader could obtain the power series expansion for $\cos^2 x$ by using the trigonometric identity $\cos^2 x \equiv \frac{1}{2}(1 + \cos 2x)$.

Example 6

Find the Maclaurin expansion of $\tanh x$ up to powers of x^5.

Solution

The first two derivatives of $f(x) = \tanh x$ are

$$f'(x) = \text{sech}^2 x \qquad f''(x) = -2\text{sech}^2 x \tanh x \qquad f'''(x) = 4\text{sech}^2 x \tanh^2 x - 2\text{sech}^4 x \quad \cdots$$

giving $\quad f(0) = 0, \quad f'(0) - 1, \quad f''(0) = 0 \quad f'''(0) = -2 \quad \cdots$

This leads directly to the Maclaurin expansion as $\qquad \tanh x = 1 - \dfrac{1}{3}x^3 + \dfrac{2}{15}x^5 \quad \cdots$

Example 7

The relationship between the wavelength, L, the wave period, T, and the water depth, d, for a surface wave in water is given by: $\qquad L = \dfrac{gT^2}{2\pi} \tanh\left(\dfrac{2\pi d}{L}\right)$

In a particular case the wave period was 10 s and the water depth was 6.1 m. Taking the acceleration due to gravity, g, as 9.81 m s^{-2} determine the wave length.

[Hint: Use the series expansion for $\tanh x$ developed in Example 6.]

Solution

Substituting for the wave period, water depth and g we get

$$L = \frac{9.81 \times 10^2}{2\pi} \tanh\left(\frac{2\pi \times 6.1}{L}\right) = \frac{490.5}{\pi} \tanh\left(\frac{12.2\pi}{L}\right)$$

The series expansion of $\tanh x$ is given by $\qquad \tanh x = x - \dfrac{x^3}{3} + \dfrac{2x^5}{15} + \cdots$

Using the series expansion of $\tanh x$ we can approximate the equation as

$$L = \frac{490.5}{\pi}\left\{\left(\frac{12.2\pi}{L}\right) - \frac{1}{3}\left(\frac{12.2\pi}{L}\right)^3 + \cdots\right\}$$

Multiplying through by πL^3 the equation becomes

$$\pi L^4 = 490.5 \times 12.2\pi L^2 - \frac{490.5}{3} \times (12.2\pi)^3$$

This equation can be rewritten as $\qquad L^4 - 5984.1L^2 + 2930198 = 0$

Solving this as a quadratic in L^2 we get $L = 74$ m.

Using Newton-Raphson iteration this can be further refined to give a wave length of 73.9 m.

3. Differentiation of Maclaurin series

We have already noted that, by the binomial series,

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots \qquad |x| < 1$$

Thus, with x replaced by $-x$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots \qquad |x| < 1$$

We have previously obtained the Maclaurin expansion of $\ln(1+x)$:

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \qquad -1 < x \leq 1$$

Now, we differentiate both sides with respect to x:

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots$$

This result matches that found from the binomial series and demonstrates that the Maclaurin expansion of a function $f(x)$ may be differentiated term by term to give a series which will be the Maclaurin expansion of $\dfrac{df}{dx}$.

As we noted in Section 16.4 the derived series will have the **same** radius of convergence as the original series.

Task

Find the Maclaurin expansion of $(1-x)^{-3}$ and state its radius of convergence.

First write down the expansion of $(1-x)^{-1}$:

Your solution

$$\frac{1}{1-x}$$

Answer

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots \qquad |x| < 1$$

Now, by differentiation, obtain the expansion of $\dfrac{1}{(1-x)^2}$:

Your solution

$$\frac{1}{(1-x)^2} = \frac{d}{dx}\left(\frac{1}{1-x}\right) =$$

Answer

$$\frac{1}{(1-x)^2} = \frac{d}{dx}(1 + x + x^2 + \cdots) = 1 + 2x + 3x^2 + 4x^3$$

Differentiate again to obtain the expansion of $(1-x)^{-3}$:

Your solution

$$\frac{1}{(1-x)^3} = \frac{1}{2}\frac{d}{dx}\left(\frac{1}{(1-x)^2}\right) = \frac{1}{2}[\qquad\qquad]$$
$$=$$

Answer

$$\frac{1}{(1-x)^3} = \frac{1}{2}\frac{d}{dx}\left(\frac{1}{(1-x)^2}\right) = \frac{1}{2}[2+6x+12x^2+20x^3+\cdots] = 1+3x+6x^2+10x^3+\cdots$$

Finally state its radius of convergence:

Your solution

Answer

The final series: $1+3x+6x^2+10x^3+\cdots$ has radius of convergence $R=1$ since the original series has this radius of convergence. This can also be found directly using the formula $R = \lim_{n\to\infty}\left|\frac{b_n}{b_{n+1}}\right|$ and using the fact that the coefficient of the n^{th} term is $b_n = \frac{1}{2}n(n+1)$.

4. The Taylor series

The **Taylor series** is a generalisation of the Maclaurin series being a power series developed in powers of $(x-x_0)$ rather than in powers of x. Thus

Key Point 15

Taylor Series

If the function $f(x)$ can be differentiated as often as required at $x=x_0$ then:

$$f(x) = f(x_0) + (x-x_0)f'(x_0) + \frac{(x-x_0)^2}{2!}f''(x_0) + \cdots$$

This is called the Taylor series of $f(x)$ about the point x_0.

The reader will see that the Maclaurin expansion is the Taylor expansion obtained if x_0 is chosen to be zero.

Task

Obtain the Taylor series expansion of $\dfrac{1}{1-x}$ about $x = 2$. (That is, find a power series in powers of $(x-2)$.)

First, obtain the first three derivatives and the n^{th} derivative of $f(x) = \dfrac{1}{1-x}$:

Your solution

$f'(x) =$ $\qquad$ $f''(x) =$ $\qquad$ $f'''(x) =$ $\qquad$ $f^{(n)}(x) =$

Answer

$$f'(x) = \frac{1}{(1-x)^2}, \qquad f''(x) = \frac{2}{(1-x)^3}, \qquad f'''(x) = \frac{6}{(1-x)^4}, \qquad \cdots \quad f^{(n)}(x) = \frac{n!}{(1-x)^{n+1}}$$

Now evaluate these derivatives at $x_0 = 2$:

Your solution

$f'(2) =$ $\qquad$ $f''(2) =$ $\qquad$ $f'''(2) =$ $\qquad$ $f^{(n)}(2) =$

Answer

$f'(2) = 1, \qquad f''(2) = -2, \qquad f'''(2) = 6, \qquad f^{(n)}(2) = (-1)^{n+1}n!$

Hence, write down the Taylor expansion of $f(x) = \dfrac{1}{1-x}$ about $x = 2$:

Your solution

$\dfrac{1}{1-x} =$

Answer

$$\frac{1}{1-x} = -1 + (x-2) - (x-2)^2 + (x-2)^3 + \cdots + (-1)^{n+1}(x-2)^n + \cdots$$

Exercises

1. Show that the series obtained in the last Task is convergent if $|x-2|<1$.

2. Sketch the linear, quadratic and cubic approximations to $\dfrac{1}{1-x}$ obtained from the series in the last task and compare to $\dfrac{1}{1-x}$.

Answer

2. In the following diagrams some of the terms from the Taylor series are plotted to compare with $\dfrac{1}{(1-x)}$.